



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

An Algebro-geometric Construction of Lower Central Series of Associative Algebras

Citation for published version:

Jordan, D & Orem, H 2015, 'An Algebro-geometric Construction of Lower Central Series of Associative Algebras', *International Mathematics Research Notices*, vol. 2015, no. 15, pp. 6330-6352.
<https://doi.org/10.1093/imrn/rnu125>

Digital Object Identifier (DOI):

[10.1093/imrn/rnu125](https://doi.org/10.1093/imrn/rnu125)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

International Mathematics Research Notices

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



An algebro-geometric construction of lower central series of associative algebras

David Jordan & Hendrik Orem

March 2, 2018

Abstract

The lower central series invariants M_k of an associative algebra A are the two-sided ideals generated by k -fold iterated commutators; the M_k provide a filtration of A . We study the relationship between the geometry of $X = \operatorname{Spec} A_{ab}$ and the associated graded components N_k of this filtration. We show that the N_k form coherent sheaves on a certain nilpotent thickening of X , and that Zariski localization on X coincides with noncommutative localization of A . Under certain freeness assumptions on A , we give an alternative construction of N_k purely in terms of the geometry of X (and in particular, independent of A). Applying a construction of Kapranov, we exhibit the N_k as natural vector bundles on the category of smooth schemes.

1 Introduction

In [Kap98], Kapranov proposes the following viewpoint on non-commutative algebraic geometry. Given a non-commutative algebra A with abelianization A_{ab} , the natural surjection $A \twoheadrightarrow A_{ab}$ can be understood as an inclusion $X = \operatorname{Spec}(A_{ab}) \hookrightarrow \operatorname{Spec}(A)$, where $\operatorname{Spec}(A)$ is the (would-be) spectrum of the algebra A . Kapranov then proposes to study so-called NC-complete algebras: the NC-completion of an algebra A is the “formal neighborhood” to the embedding $X \hookrightarrow \operatorname{Spec}(A)$; thus considering NC-complete algebras A amounts to studying formal thickenings of X into a non-commutative scheme.

A basic invariant of a non-commutative algebra A is its lower central series, the descending filtration $L_\bullet = L_\bullet(A)$ by Lie ideals, defined recursively as follows:

$$L_1 := A, \quad L_k := [A, L_{k-1}].$$

In other words, L_k is spanned by iterated commutators, $[a_1, [a_2, [\dots [a_{k-1}, a_k] \dots]]$, for $a_i \in A$. The lower central series ideals are $M_k := AL_kA = AL_k$, and the associated graded components are $N_k := M_k/M_{k+1}$. In particular, we have $N_1 = A_{ab} = \mathcal{O}(X)$. In fact, the components N_k depend only on the NC-completion of A , so they are really invariants of Kapranov’s formal neighborhood of X in A , rather than of A itself.

For several years, evidence has been mounting that – despite their origin in non-commutative algebra – the associated graded components N_k , and their Lie analogs $B_k := L_k/L_{k+1}$, should

admit descriptions in terms of the *commutative* algebraic geometry of $X = \text{Spec}(A/M_2)$. The first such evidence was provided by Feigin and Shoikhet [FS07] for the free algebra, A_n , on n generators. They constructed an isomorphism $A_n/M_3 \cong \Omega^{ev}(\mathbb{A}^n)$, identifying the first two components $N_1 \oplus N_2$ with the space of even-degree differential forms on $X = \mathbb{C}^n$, with Fedosov quantized product (see Example 2.10). In [DE08] and [EKM09], it is shown more generally that the components $B_k(A_n)$ and $N_k(A_n)$, respectively, are finite extensions of tensor field modules on \mathbb{A}^n , with respect to a natural action of the Lie algebra W_n of polynomial vector fields. The papers [AJ10], [BJ13a], [BB11], [BEJ⁺12], [BJ13b], [Ker13], all apply geometric techniques echoing [FS07] to compute $B_k(A)$ and $N_k(A)$ in various special cases.

The present paper aims to explain the geometric nature of the lower central series filtration in two essential ways: Firstly, we prove that the components $N_k(A)$ are Zariski local on X , and are in fact coherent sheaves on a certain canonical nilpotent thickening of X . Secondly, under the assumption that A is locally free – in particular, for Kapranov’s NC-manifolds, and for formally smooth (a.k.a quasi-free) algebras of Cuntz-Quillen [CQ95] – we give an alternative construction of the lower central series in terms of the formal geometry of X . This construction implies that $N_k(A)$ is not only local on X , but it is in fact formal: $N_k(A)$ is completely determined by its value in a formal neighborhood of any point $x \in X$, which in turn has a universal answer depending only on the dimension of X .

An outline of the paper is as follows. In Section 2.2, for an arbitrary finitely generated algebra A , we begin by introducing a commutative product on the quotient A/M_3 as follows:

Proposition 2.6. The star-product, $a \star b := \frac{1}{2}(ab + ba)$, is associative and commutative on $A_\star := (A/M_3, \star)$, and makes each $N_k(A)$ into a finitely generated A_\star -module.

Kapranov has shown in [Kap98] that NC-complete algebras A admit flat Ore localizations with respect to multiplicative subsets $S \subset A/M_2$ without zero divisors. Our first main result – the content of Section 3 – is that Ore localization of an algebra A coincides with the ordinary, *commutative* localization on the A_\star -module $N_k(A)$:

Theorem 3.1. Fix a multiplicative subset $\bar{S} \subset (A/M_2)$ without zero divisors, and let $S := \pi^{-1}(\bar{S}) \subset A$. The map,

$$m : A_\star[S^{-1}] \otimes_{A_\star} N_k(A) \xrightarrow{\sim} N_k(A[S^{-1}]),$$

$$a \otimes n \mapsto an,$$

is an isomorphism.

The significance of this theorem is that it allows us to work with $N_k(A)$ as a coherent sheaf $\widetilde{N}_k(A)$ on X_\star , applying tools from commutative algebraic geometry, such as localizations, completions, and Hilbert’s syzygy theorem, to perform computations. Let \widehat{A}_n denote the degree completion of the free algebra A_n . As applications of Theorem 3.1 we have:

Corollary 3.9. The completion $N_k(A)_{(\mathfrak{m}_x)}$ at any smooth point $x \in X^{sm}$ is a quotient of $N_k(\widehat{A}_n)$, where $n = \dim X$.

Corollary 3.10. Suppose that $X = \text{Spec}(A/M_2)$ is a zero or one dimensional scheme with finitely many non-reduced points. Then each $N_k(A)$ is finite dimensional, for $k \geq 2$.

Corollary 3.12. Suppose A is the quotient of a free algebra on generators x_1, \dots, x_n in degrees d_1, \dots, d_n , by a collection of homogeneous non-commutative polynomials, so that A is graded by degree. Then the Hilbert series for each $N_k(A)$ is a rational function, with poles at d_i th roots of unity.

In Section 4, we turn our attention to locally free algebras.

Definition 1.1. An algebra A is called *locally free of rank n* if, for every $x \in X$, there exists an isomorphism $A_{(\mathfrak{m}_x + M_2)} \cong \hat{A}_n$.

Example 1.2. There are many examples of locally free algebras:

- (i) The free algebra A_n is locally free of rank n .
- (ii) The coordinate algebra $\mathcal{O}(X)$ of any connected smooth affine curve X is locally free of rank one.
- (iii) The free product of locally free algebras of ranks m and n is locally free of rank $m + n$.
- (iv) For any smooth complete intersection $X = \mathbb{C}[x_1, \dots, x_{n+m}]/\langle f_1, \dots, f_m \rangle$, the algebra $A = A_{n+m}/\langle \tilde{f}_1, \dots, \tilde{f}_m \rangle$, where each $\tilde{f}_i \in f_i + M_2$, is locally free of rank $\dim X = n$.
- (v) If A is locally free, and if \bar{S} is any multiplicative subset of A/M_2 without zero divisors, then $A[S^{-1}]$ is locally free, where $S = \pi^{-1}(\bar{S})$.
- (vi) Any NC-smooth algebra (see [Kap98], Proposition 1.5.1), including Kapranov's NC-smooth thickenings of any finitely generated smooth commutative algebra (see [Kap98], Proposition 1.6.1), is locally free.
- (vii) Any formally smooth (a.k.a quasi-free) algebra is locally free, by the formal tubular neighborhood theorem ([CQ95], Section 6, Theorem 2).

Remark 1.3. We have isomorphisms:

$$(A/M_2)_{(\mathfrak{m}_x)} \cong A_{(\mathfrak{m}_x + M_2)}/M_2 \cong \hat{A}_n/M_2 \cong \mathbb{C}[[x_1 \dots x_n]].$$

Thus if A is locally free of rank n , then X is smooth of dimension n .

Under these assumptions, we give a completely new construction of $N_k(A)$ in the language of formal geometry on X . Given a module M over the Lie algebra W_n , which satisfies certain finiteness conditions, we have the globalization $\mathcal{GL}_X(M)$, a sheaf on X which “spreads” M around X , using the W_n -action to produce transition functions (see Section 4.3 for a review of formal geometry). Our second main result is:

Theorem 4.26. We have $\widetilde{N}_k(A) \cong \mathcal{GL}_X(N_k(\hat{A}_n))$ as vector bundles on X . In particular, we have $N_k(A) \cong \Gamma(X, \mathcal{GL}_X(N_k(\hat{A}_n)))$, as $U(\text{Vect}(X)) \ltimes O(X)$ -modules.

Combining Theorems 3.1 and 4.26, we exhibit N_k as a *natural vector bundle* - a functorial assignment to any smooth scheme X of a vector bundle over X ; remarkably this characterization is completely independent of the choice of locally free lift A . It is well known that the category of natural vector bundles is equivalent to the category of finite dimensional representations of vector fields on a formal disc vanishing at the origin; this allows us to reduce the problem of describing the components N_k for all locally free algebras of a given rank to a single, finite dimensional problem in representation theory; some example computations are given at the end of Section 4.3.

Acknowledgments

The existence of a description of the lower central series ideals via formal geometry was conjectured by Pavel Etingof. We are grateful to him for sharing his conjecture with us, and for many helpful comments and suggestions as the paper progressed. We are also grateful to David Ben-Zvi for many explanations concerning Fedosov quantization, formal geometry and the geometric interpretation of [Kap98].

2 Preliminaries

For the remainder of the paper, A denotes a finitely generated associative algebra, and $X := \text{Spec}(A/M_2(A))$.

2.1 NC-nilpotent algebras and localizations

In this subsection, we compare the filtration of A by lower central series ideals to Kapranov's NC-filtration [Kap98].

Definition 2.1. We consider the following filtrations on A :

1. The lower central series filtration L_\bullet is defined recursively by $L_1 = A, L_k := [A, L_{k-1}]$.
2. The associative lower central series filtration M_\bullet is defined by $M_k := AL_kA = AL_k$.
3. The NC-filtration F_\bullet is defined by $F_k = \sum_{i_1+\dots+i_m=k-m} M_{i_1} \cdots M_{i_m}$.

We denote by $B_k := L_k/L_{k+1}, N_k := M_k/M_{k+1}$ the associated graded components of each filtration.

Remark 2.2. In Sections 3 and 4 we apply geometric techniques to the graded components of the filtration M_k . The same results in fact hold for the associated graded components of F_k , with easier proofs (compare, for example, Theorem 3.1 with Theorem 2.1.6 of [Kap98]).

Definition 2.3. An algebra A is *Lie complete* (resp, *NC-complete*) if it is complete in the topology induced by the filtration M_\bullet (resp, F_\bullet). Similarly, A is *Lie nilpotent* (resp, *NC-nilpotent*) if $M_N(A) = 0$ (resp, $F_N(A) = 0$) for some N .

Clearly, we have that $M_k \subset F_{k-1}$. Conversely, ([Jen47], Theorem 2) proves that for any finitely generated algebra, and any k , there exists N such that $M_2^N \subset M_k$. In particular, $F_N \subset M_k$. Thus we have:

Claim 2.4. *A finitely generated algebra A is NC-complete if, and only if, it is Lie complete. Similarly, A is NC-nilpotent if and only if it is Lie nilpotent.*

Following [Kap98], we will use the term NC-nilpotent for what is called Lie nilpotent in [EKM09].

Proposition 2.5. *Suppose A is NC-nilpotent. Let $\overline{S} \subset (A/M_2)$ be a finitely generated multiplicative set without zero divisors, and define $S = \pi^{-1}(\overline{S})$, where $\pi : A \rightarrow A/M_2$ is the quotient map. Then:*

- (1) S satisfies the Ore condition.
- (2) $A[S^{-1}]$ is finitely generated.
- (3) $A[S^{-1}]$ is NC-nilpotent.

Proof. Claim (1) is precisely Proposition 2.1.5 in [Kap98]. By Claim 2.4, F_\bullet and M_\bullet induce the same completion, and hence A is nilpotent for one filtration if and only if it is nilpotent for the other. Claim (3) thus follows from Theorem 2.1.6 in [Kap98].

To establish finite generation, we show that $(S + M_2)^{-1}$ is generated by A and S^{-1} . For $s \in S$ and $m \in M_2$, we express $(s + m)^{-1}$ as a geometric series,

$$\begin{aligned} (s + m)^{-1} &= s^{-1}(1 + s^{-1}m)^{-1} \\ &= s^{-1} \left(\sum_{i \geq 0} (-s^{-1}m)^i \right), \end{aligned}$$

which is seen to be a finite sum using the NC-nilpotence of A , and identity (2.1.7.2) in [Kap98]:

$$ms^{-1} = s^{-1}m + s^{-2}[s, m] + s^{-3}[s, [s, m]] + \cdots$$

□

2.2 The algebra A_\star

The algebra A/M_3 carries a commutative multiplication, in addition to the non-commutative multiplication induced from A . We have:

Proposition 2.6. *The star-product, $a \star b := \frac{1}{2}(ab + ba)$, is associative on $A_\star := (A/M_3, \star)$, and the action $a \star n := \frac{1}{2}(an + na)$, for $a \in A_\star, n \in N_k(A)$ makes each $N_k(A)$ into an A_\star -module.*

The proof of Proposition 2.6 depends on the following two lemmas:

Lemma 2.7 (Corollary 6.4 in [BJ13a]). *If i is odd, then $M_i M_j \subset M_{i+j-1}$.*

Lemma 2.8. *We have the containment $[A, [A, M_k]] \subset M_{k+1}$.*

Proof. We let $a, b, c \in A, d \in L_k$, and apply Leibniz identity:

$$[a, [b, cd]] = [a, [b, c]d] + [a, c[b, d]] = [a, [b, c]]d + [b, c][a, d] + [a, c[b, d]].$$

Applying Lemma 2.7, we have that all three terms on the right are in M_{k+1} . \square

Proof of Proposition 2.6. To see that \star is associative, we compute:

$$(a \star b) \star c - a \star (b \star c) = \frac{1}{4} ((ab + ba)c + c(ab + ba) - a(bc + cb) - (bc + cb)a) = \frac{1}{4} [b, [a, c]],$$

and thus associativity holds, modulo M_3 .

The containment $M_3 M_k \subset M_{k+2}$ of Lemma 2.7 shows the map of vector spaces $A/M_3 \otimes N_k \rightarrow N_k$ is well-defined. Associativity of the action follows from Lemma 2.8; for $n \in M_k$, we have:

$$(a \star b) \star n - a \star (b \star n) = \frac{1}{4} [b, [a, n]] = 0 \pmod{M_{k+1}}.$$

\square

Definition 2.9. We denote by A_\star the algebra $(A/M_3, \star)$, and we let $X_\star := \text{Spec} A_\star$.

Example 2.10. When $A = A_n$, the Feigin-Shoikhet isomorphism [FS07] identifies A/M_3 with the algebra $\Omega^{ev}(\mathbb{C}^n)$ of even degree differential forms on \mathbb{C}^n , equipped with the Fedosov quantized product,

$$\omega \nu := \omega \wedge \nu + d\omega \wedge d\nu.$$

In this case A_\star is the commutative algebra of even-degree differential forms, with ordinary product.

3 Lower central series components as coherent sheaves on X_\star

Throughout this section we assume that A is an NC-nilpotent algebra, with $M_N(A) = 0$. In the previous section, we endowed each $N_k(A)$ with the structure of a module for the commutative algebra A_\star , and hence obtained a quasi-coherent sheaf \widetilde{N}_k on X_\star . On the other hand, N_k also determines a pre-sheaf on X_\star by the assignment on affine opens $U_{\bar{S}} \subset X_\star \mapsto N_k(A[S^{-1}])$. The purpose of this section is to show that the two pre-sheaves obtained this way coincide; this is the content of:

Theorem 3.1. *Fix a multiplicative subset $\bar{S} \subset (A/M_2)$ without zero divisors, and let $S := \pi^{-1}(\bar{S}) \subset A$. The map,*

$$m : A_\star[S^{-1}] \otimes_{A_\star} N_k(A) \xrightarrow{\sim} N_k(A[S^{-1}]),$$

$$a \otimes n \mapsto an,$$

is an isomorphism.

Remark 3.2. It follows from [EKM09], Theorem 2.1, that $N_k(A)$ is finitely generated over A/M_3 in the case that A is a free algebra of finite rank, and thus whenever A is finitely generated. Thus each $N_k(A)$ is a coherent sheaf over X_\star .

Remark 3.3. We note that there is no loss of generality in assuming A is NC-nilpotent: we have $N_k(A) \cong N_k(A/M_{k'})$ for $k' > k$; this allows us to apply Theorem 3.1 to an arbitrary finitely generated algebra by first quotienting by sufficiently large $M_{k'}$.

The remainder of this section is devoted to proving Theorem 3.1. We will make repeated use of the following computation:

Lemma 3.4. *Let $f \in A$ invertible. Then we have: $[f^{-1}, g] = -f^{-2}[f, g] \pmod{M_3(A)}$.*

Proof. We compute:

$$\begin{aligned} 0 &= f^{-1}[f^{-1}f, g] \\ &= f^{-1}(f^{-1}[f, g] + [f^{-1}, g]f) \\ &= f^{-1}(f^{-1}[f, g] + f[f^{-1}, g] + [[f^{-1}, g], f]) \\ &= f^{-2}[f, g] + [f^{-1}, g] + f^{-1}[[f^{-1}, g], f]. \end{aligned}$$

□

Remark 3.5. In the Feigin-Shoikhet isomorphism $A/M_3 \cong \Omega^{ev}(\mathbb{C}^n)$, a commutator $[f, g]$ is sent to the two-form $2df \wedge dg$. We note that Lemma 3.4 is compatible with the Feigin-Shoikhet isomorphism, via the identity $d(\frac{1}{f}) = \frac{-1}{f^2}df$.

For elements $a_1, \dots, a_l \in A$, we denote by $M_k(a_1, \dots, a_l)$ the image of $M_k(A_l)$ under the homomorphism $A_l \rightarrow A$, given by $x_i \mapsto a_i$.

Lemma 3.6. *For $m \in M_k$, the expression*

$$[b, d][a, m] + [a, d][b, m] \tag{1}$$

lies in M_{k+2} .

Proof. The expression maps to zero under the Feigin-Shoikhet map, so it lies in $M_3(a, b, d, m)$. The space $M_3(a, b, d, m)$ is spanned by the elements:

$$m[a, [b, d]], a[b, [d, m]], [a, [b, dm]], [ab, [d, m]], [a, [bd, m]],$$

together with their permutations in the symbols a, b, d . We have $m[a, [b, d]] \in M_k M_3 \subset M_{k+2}$, by Lemma 2.7 while the remaining elements lie in M_{k+2} , by Lemma 2.8. □

Theorem 3.7. *For k odd, we have:*

$$[A, [A, S^{-1}M_k(A)]] \subset S^{-1}M_{k+2}(A) + M_{k+3}(A[S^{-1}]).$$

Proof. Let $T = S^{-1}M_{k+2}(A) + M_{k+3}(A[S^{-1}])$. We need to show that, for $a, b \in A, d \in S$, and $m \in M_k(A)$, we have $[a, [b, d^{-1}m]] = 0 \pmod T$. The proof consists of moving d^{-1} out of the nested commutator, modulo terms in $M_{k+3}(A[S^{-1}])$. We compute:

$$[a, [b, d^{-1}m]] = [a, [b, d^{-1}]m] + [a, d^{-1}[b, m]] \quad (2)$$

$$= -[a, d^{-2}[b, d]m] + [a, d^{-1}[d^{-1}, [b, d]]m] + [a, d^{-1}][b, m] + d^{-1}[a, [b, m]] \quad (3)$$

$$= -[a, d^{-2}[b, d]m] + [a, d^{-1}][b, m] + d^{-1}[a, [b, m]] \pmod{L_{k+3}(A[S^{-1}])} \quad (4)$$

In passing from equation (2) to (3), we have manipulated the first term as in the proof of Lemma 3.4, and applied Jacobi identity to the second term. Because k is odd, Theorem 1.3 of [BJ13a] implies that the second term in (3) lies in $[A, M_3M_k] \subset [A, M_{k+2}] \subset L_{k+3}(A[S^{-1}])$, so it is zero modulo T . The third summand in equation (4) lies in T , so may be ignored. Expanding the first summand of equation (4) with the Leibniz rule, and applying Lemma 3.4 to the second, we obtain:

$$[a, [b, d^{-1}m]] = -d^{-2}[a, [b, d]m] - [a, d^{-2}][b, d]m - d^{-2}[a, d][b, m] \pmod T. \quad (5)$$

Applying Lemma 3.4 to the second RHS term of (5) gives:

$$2d^{-3}[a, d][b, d]m \in S^{-1}M_3(A)M_k(A) \subset S^{-1}M_{k+2}(A) \subset T,$$

as in the proof of Lemma 3.6. Thus we have:

$$\begin{aligned} [a, [b, d^{-1}m]] &= -d^{-2}([a, [b, d]m] + [a, d][b, m]) \pmod T \\ &= -d^{-2}([b, d][a, m] + [a, d][b, m]) - d^{-2}[a, [b, d]]m \pmod T, \end{aligned}$$

by the Jacobi identity. By Lemma 3.6 the first expression lies in $S^{-1}M_{k+2}(A) \subset T$; by Lemma 2.7, the second expression does as well. \square

Corollary 3.8. *We have:*

1. $M_k(A[S^{-1}]) = A[S^{-1}] \otimes_A M_k(A) + M_{k+1}(A[S^{-1}])$.
2. $M_k(A[S^{-1}]) = A[S^{-1}] \otimes_A M_k(A)$.

Proof. Assertion (2) follows by repeatedly applying (1), and using that $A[S^{-1}]$ is NC-nilpotent. For (1), we consider an arbitrary generator of $M_k(A[S^{-1}])$,

$$v = a[l_1, [l_2, [\dots [l_{k-1}, l_k] \dots]],$$

where the l_i are some monomials in the generators of $A[S^{-1}]$. If each l_i lies in A , there is nothing to prove; thus, we suppose that $l_i = \tilde{l}_i f^{-1}$, for some $\tilde{l}_i \in A$, $f \in S$, and $i \in I$. By repeated use of Jacobi identity, we may assume that $i = k$.

By repeatedly applying Theorem 3.7, we may pull the f^{-1} out of the nested commutator two slots at a time, until we are left with expressions of the form:

$$\begin{aligned} &af^{-1}[l'_1, [l'_2, \dots [l'_{k-1}, l'_k] \dots]], \text{ if } k \text{ is odd, or} \\ &a[l'_1, f^{-1}[l'_2, \dots [l'_{k-1}, l'_k] \dots]], \text{ if } k \text{ is even,} \end{aligned}$$

where we have ignored any additional terms produced which lie in $M_{k+1}(A[S^{-1}])$. In the first case, there is one less inversion in the iterated commutator expression, so by induction v lies in $A[S^{-1}] \otimes_A M_k(A) + M_{k+1}(A[S^{-1}])$. In the second case, we compute, using the Jacobi identity:

$$\begin{aligned} & a[l'_1, f^{-1}[l'_2, \dots [l'_{k-1}, l'_k] \dots]] \\ &= af^{-1}[l'_1, [l'_2, \dots [l'_{k-1}, l'_k] \dots]] + a[l'_1, f^{-1}][l'_2, \dots [l'_{k-1}, l'_k] \dots] \\ &= af^{-1}[l'_1, [l'_2, \dots [l'_{k-1}, l'_k] \dots]] - af^{-2}[l'_1, f][l'_2, \dots [l'_{k-1}, l'_k] \dots] \pmod{M_{k+1}(A[S^{-1}])}. \end{aligned}$$

The second summand above lies in $M_2 M_{k-1} \subset M_k$, and both summands feature one less inversion in the nested commutator, and so by induction lie in $A[S^{-1}] \otimes_A M_k(A) + M_{k+1}(A[S^{-1}])$. \square

Proof of Theorem 3.1. Having proved Corollary 3.8, the proof echoes Theorem 2.1.6 of [Kap98]. Namely, we have an exact sequence:

$$0 \rightarrow M_{k+1} \rightarrow M_k \rightarrow N_k \rightarrow 0. \quad (6)$$

By Proposition 2.5, $A[S^{-1}]$ is a flat A -module; applying $A[S^{-1}] \otimes_A -$ we obtain:

$$0 \rightarrow A[S^{-1}] \otimes_A M_{k+1} \rightarrow A[S^{-1}] \otimes_A M_k \rightarrow A[S^{-1}] \otimes_A N_k \rightarrow 0. \quad (7)$$

Combining this with Corollary 3.8, we obtain

$$A[S^{-1}] \otimes_A N_k(A) \cong \frac{A[S^{-1}] \otimes_A M_k}{A[S^{-1}] \otimes_A M_{k+1}} \cong \frac{M_k(A[S^{-1}])}{M_{k+1}(A[S^{-1}])} \cong N_k(A[S^{-1}]). \quad (8)$$

\square

Corollary 3.9. *The completion $N_k(A)_{(\mathfrak{m}_x)}$ at any smooth point $x \in X^{sm}$ is a quotient of $N_k(\widehat{A}_n)$, where $n = \dim X$.*

Corollary 3.10. *Suppose that $X = \text{Spec}(A/M_2)$ is a zero or one dimensional scheme with finitely many non-reduced points. Then each $N_k(A)$ is a finite dimensional vector space, for $k \geq 2$.*

Proof. By Theorem 3.1, each $N_k(A)$ forms a coherent sheaf on X_\star . Over any smooth, reduced point of X , the fiber is isomorphic to $N_k(A_n) = 0$, for $n = 0, 1$; hence the sheaf is supported only over the singular and non-reduced points of X , where it is of finite rank. \square

Remark 3.11. The assumption that there are finitely many non-reduced points is necessary. For example, let $A = k\langle x, y \rangle / (y^2)$. Then the set $\{[x^l, y], l \geq 1\}$ is linearly independent in N_2 . See [CF13] for an elaboration of this example, and related examples.

Corollary 3.12. *Suppose A is the quotient of a free algebra on generators x_1, \dots, x_n in degrees d_1, \dots, d_n , by a collection of homogeneous non-commutative polynomials, so that A is graded by degree. Then the Hilbert series for each $N_k(A)$ is a rational function, with poles at d_i th roots of unity.*

Proof. This follows immediately from Hilbert's syzygy theorem for graded modules over polynomial rings. \square

4 Formal geometry and locally free algebras

Throughout this section, we assume A is a finitely generated, locally free algebra of rank n . We begin with a review of the basic constructions in formal geometry, and then give an alternative construction of $N_k(A)$ in terms of the formal geometry on X . For background, we refer to [BK04], Section 3, whose conventions we follow.

4.1 Harish-Chandra torsors

Let G be a pro-algebraic group, $G = \varprojlim G^k$. Let $\mathfrak{g} = \text{Lie}(G)$ denote the Lie algebra of G , and let $\rho : \mathcal{M} \rightarrow X$ be a G torsor, that is, a scheme \mathcal{M} over X , equipped with a G action commuting with ρ , and inducing an isomorphism $G \times \mathcal{M} \cong \mathcal{M} \times_X \mathcal{M}$.

Definition 4.1. We say a G -module N is a *pro-finite* if we have $N = \varprojlim N^k$ of finite dimensional representations N^k of the algebraic groups G^k .

Throughout this section, we take all G -modules to be pro-finite.

Definition 4.2. Given a G -module N , the *associated vector bundle* $N_{\mathcal{M}}$ on X is:

$$N_{\mathcal{M}} := \mathcal{M} \times N / \{(m \cdot g, v) \sim (m, g \cdot v)\}. \quad (9)$$

Remark 4.3. By construction, the fiber of $N_{\mathcal{M}}$ over $x \in X$ is non-canonically isomorphic to N , and we have an isomorphism,

$$\Gamma(X, N_{\mathcal{M}}) \cong \Gamma(\mathcal{M}, \mathcal{M} \times N)^G. \quad (10)$$

We have an associated G -equivariant exact sequence of sheaves on \mathcal{M} :

$$0 \rightarrow \mathcal{T}_{\mathcal{M}/X} \rightarrow \mathcal{T}_{\mathcal{M}} \rightarrow \rho^* \mathcal{T}_X \rightarrow 0 \quad (11)$$

which, by descent gives us an exact sequence of sheaves on X (known as the *Atiyah extension*)

$$0 \rightarrow \mathfrak{g}_{\mathcal{M}} \xrightarrow{i} \mathcal{E}_{\mathcal{M}} \xrightarrow{j} \mathcal{T}_X \rightarrow 0. \quad (12)$$

Here $\mathfrak{g}_{\mathcal{M}}$ denotes the bundle associated to the G -module \mathfrak{g} , as explained above. Recall that a connection on \mathcal{M} is a splitting $\theta_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{g}_{\mathcal{M}}$. We will need to consider a more general notion of connection, taking values in a larger Lie algebra than \mathfrak{g} .

Definition 4.4. A *Harish-Chandra pair* $\langle G, \mathfrak{h} \rangle$ is a pro-algebraic group G , together with a Lie algebra \mathfrak{h} with G -action, and a compatible inclusion $\iota : \mathfrak{g} \hookrightarrow \mathfrak{h}$.

Definition 4.5. A connection $\theta_{\mathcal{M}}$ for $\langle G, \mathfrak{h} \rangle$ is a map $\theta_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$ such that $\iota = \theta_{\mathcal{M}} \circ i$. The pair $(\mathcal{M}, \theta_{\mathcal{M}})$ is called a $\langle G, \mathfrak{h} \rangle$ -torsor with connection.

Definition 4.6. For G connected, we say that an \mathfrak{h} -module N is a $\langle G, \mathfrak{h} \rangle$ -module if the induced action of \mathfrak{g} is integrated to an algebraic representation of G .

Given a $\langle G, \mathfrak{h} \rangle$ -torsor with connection $(\mathcal{M}, \theta_{\mathcal{M}})$ a $\langle G, \mathfrak{h} \rangle$ -module N , the vector bundle $N_{\mathcal{M}}$ carries a connection ∇ , called the Harish-Chandra connection, given by the formula, for $s \in \Gamma(U, N_{\mathcal{M}})$ and $\xi \in \Gamma(U, \mathcal{T}_X)$:

$$\nabla_{\xi}(s) := \tilde{\xi} \cdot s - \theta_{\mathcal{M}}(\tilde{\xi}) \cdot s,$$

where $\tilde{\xi}$ is any lift of ξ to $\mathcal{E}_{\mathcal{M}}$; in fact ∇_{ξ} is independent of the choice of lift.

Remark 4.7. Given a pair $\langle G, \mathfrak{h} \rangle$, one can define a formal group H which contains G and has Lie algebra \mathfrak{h} such that H/G is the formal neighborhood of zero in $\mathfrak{h}/\mathfrak{g}$. Then there is an H -torsor associated to any G -torsor, and splitting an analogue of (12) for this H -torsor is equivalent to a connection on a $\langle G, \mathfrak{h} \rangle$ -torsor as defined above.

4.2 Formal geometry

The fundamental example of a Harish-Chandra torsor we will use is the *bundle of formal coordinate systems*, whose construction we now recall.

Let D_n denote the formal disc, $D_n = \text{spf}(\mathbb{C}[[x_1, \dots, x_n]])$; denote the unique maximal ideal by \mathfrak{m}_0 . We have the pro-algebraic group $\text{Aut}(D_n) = \varprojlim \text{Aut}^k(D_n)$, the inverse limit over k of the automorphism groups $\text{Aut}^k(D_n)$ of k -th order infinitesimal neighborhoods of the origin in \mathbb{A}^n .

Let W_n denote the Lie algebra of vector fields on D_n , and let W_n^0 denote the subalgebra of vector fields vanishing at the closed point and more generally let W_n^k denote vector fields vanishing to order $k+1$ at the closed point. In other words, W_n is the Lie algebra of derivations of $\mathbb{C}[[x_1, \dots, x_n]]$, W_n^0 is the subalgebra preserving the augmentation ideal, and W_n^k is the subalgebra $\mathfrak{m}_0^{k+1}W_n$, under the natural $\mathbb{C}[[x_1, \dots, x_n]]$ -module structure on W_n . We have an identification $W_n^0 = \text{Lie}(\text{Aut}(D_n))$.

The following proposition is well known.

Proposition 4.8. *A finite dimensional W_n^0 -module N integrates to an $\text{Aut}(D_n)$ -module if, and only if, the Euler operator $E = \sum x_i \partial_i$ acts diagonalizably with integer eigenvalues. Moreover, for such N there exists k such that the $\text{Aut}(D_n)$ -action on N factors through $\text{Aut}^k(D_n)$.*

Using the above proposition it is easy to confirm that all W_n -modules encountered in this section give rise to $\langle \text{Aut}(D_n), W_n \rangle$ -modules.

Example 4.9. Examples of pro-finite representations of $\text{Aut}(D_n)$ coming from geometry include power series rings $M = k[[x_1, \dots, x_n]]$, and more generally, the completion of a natural vector bundle at a smooth point on a variety of dimension n (see Section 4.3).

Definition 4.10. The *bundle of formal coordinate systems* \mathcal{M}_X on a smooth scheme X of dimension n consists of pairs $(x \in X, t_x : X_{(x)} \xrightarrow{\sim} D_n)$. Projection to the first factor makes \mathcal{M}_X a $\text{Aut}(D_n)$ -torsor over X .

Remark 4.11. This definition is incomplete, in that it doesn't define \mathcal{M}_X as a scheme: in fact, \mathcal{M}_X has the structure of a scheme of infinite type over X , and is locally trivial as a

$\text{Aut}(D_n)$ -torsor in the Zariski topology on X . We refer the reader to [FBZ01], Sections 9.4.4 and 11.3.3, and [BK04], Section 3.1, for complete definitions regarding the scheme structure on \mathcal{M}_X .

The action of W_n on $\mathbb{C}[[x_1, \dots, x_n]]$ defines a map,

$$\tilde{a} : W_n \times \mathcal{M}_X \rightarrow T_{\mathcal{M}_X/X},$$

descending to an isomorphism,

$$a : (W_n)_{\mathcal{M}} \xrightarrow{\sim} \mathcal{E}_{\mathcal{M}}.$$

Definition 4.12. The *bundle of flat formal coordinate systems* on X is the $\langle \text{Aut}(D_n), W_n \rangle$ -torsor, $(\mathcal{M}_X, \theta_{\mathcal{M}_X} = a^{-1})$.

Remark 4.13. Given a $\langle \text{Aut}(D_n), W_n \rangle$ -module N , and $U \subset X$ over which $N_{\mathcal{M}_X}$ trivializes, the Harish-Chandra connection $N_{\mathcal{M}_X}|_U = N_{\mathcal{M}_U}$ is given by:

$$\nabla_{\xi}(s) = ds(\xi) - \xi \circ s.$$

Thus a section $s \in \Gamma(X, N_{\mathcal{M}_X})$ is flat if the derivative, $\xi(s)$, of s along the base is simply the action of ξ in the fiber.

Definition 4.14. The globalization $\mathcal{GL}_X(N)$ to X of a $\langle \text{Aut}(D_n), W_n \rangle$ -module N is the sheaf of flat sections of the associated bundle $(N_{\mathcal{M}_X}, \nabla)$.

4.3 Natural vector bundles

A natural vector bundle is essentially a functorial assignment of a vector bundle $\mathcal{V}(X) \rightarrow X$, to every smooth X . More precisely, we have:

Definition 4.15. The category Sm_n has as objects smooth, finite-type formal schemes, of dimension n , and as morphisms étale maps.

Remark 4.16. Thus, the category Sm_n is just general enough to encompass both ordinary smooth algebraic varieties, as well as formal neighborhoods of smooth points.

Definition 4.17. The category VB_n , of vector bundles with n -dimensional base, has as objects pairs (X, V) of a smooth, finite-type formal scheme X of dimension n , and a vector bundle $V \rightarrow X$ over X . A morphism from (X, V) to (Y, W) is an étale map $f : X \rightarrow Y$, together with an isomorphism $\phi : V \rightarrow f^*(W)$.

We have a forgetful functor $F : VB_n \rightarrow Sm_n$, forgetting the vector bundle.

Definition 4.18. A natural vector bundle \mathcal{V} on n -dimensional schemes is a functorial splitting of F , i.e. a functor $G : Sm_n \rightarrow VB_n$, such that $G \circ F = \text{id}_{Sm_n}$. A morphism of natural vector bundles is a natural transformation of the corresponding functors. This defines the category, \mathcal{NVB}_n , of natural vector bundles.

Semi-simple examples of natural vector bundles include: trivial vector bundles, tangent and cotangent bundles, differential forms, polyvector fields, and more generally any tensor bundle built from tangent and cotangent bundles by Schur functors. Non-semi-simple examples include bundles of polynomial differential operators of order $\leq k$, for any k .

Definition 4.19. Let $i_0 : \{0\} \hookrightarrow D_n$ denote the inclusion of the origin into D_n . The *standard fiber* $St(\mathcal{V})$ of $\mathcal{V} \in \mathcal{NVB}_n$ is $St(\mathcal{V}) = i_0^*(\mathcal{V}(D_n))$.

Applying Definition 4.18, each automorphism $g : D_n \rightarrow D_n$ induces an isomorphism $\phi_g : \mathcal{V}(D_n) \rightarrow g^*\mathcal{V}(D_n)$. Since g preserves the closed point, the assignment $g \mapsto \phi_g$ defines an action of $Aut(D_n)$ on $St(\mathcal{V})$. Thus we have the *standard fiber* functor:

$$St : \mathcal{NVB}_n \rightarrow Aut(D_n)\text{-mod}_f,$$

to the category of finite dimensional algebraic $Aut(D_n)$ -modules.

Conversely, given a finite dimensional algebraic $Aut(D_n)$ -module N , the assignment to each smooth scheme X of its associated bundle $N_{\mathcal{M}_X}$ determines a natural vector bundle, $Assoc(N)$.

Proposition 4.20. *The functors St and $Assoc$ are mutually inverse equivalences of categories.*

Proof. Let $M \in Aut(D_n)\text{-mod}_f$. We have a natural isomorphism $M \xrightarrow{\sim} St \circ Assoc(M)$, which simply identifies M with the fiber of the associated bundle over the closed point.

For $\mathcal{V} \in \mathcal{NVB}_n$, we now construct a natural isomorphism $\mathcal{V} \rightarrow Assoc(St(\mathcal{V}))$. Set $\mathcal{G} = Assoc \circ St(\mathcal{V})$. Given $X \in Sm_n$, the data of a map $\Psi : \mathcal{G}(X) \rightarrow \mathcal{V}(X)$ is equivalent to that of a G -equivariant map $\tilde{\Psi}$ as shown:

$$\begin{array}{ccccc} \mathcal{M}_X & & \pi^*\mathcal{G}(X) & \xrightarrow{\tilde{\Psi}} & \pi^*\mathcal{V}(X) \\ \downarrow \pi & & \downarrow & & \downarrow \\ X & & \mathcal{G}(X) & \xrightarrow{\Psi} & \mathcal{V}(X) \end{array}$$

with dashed arrows indicating the descent functor (i.e. taking $Aut(D_n)$ -invariants). Note that any point $(x, \varphi) \in \mathcal{M}_X$ gives a map $\varphi_x : St(\mathcal{V}) \xrightarrow{\sim} \mathcal{V}_x$, the fiber of \mathcal{V} at $x \in X$. Now let s be a section of $\pi^*\mathcal{G}(X)$ over \mathcal{M}_X . We define:

$$\left(\tilde{\Psi}(s)\right)(x, \varphi) := (x, \varphi_x(s(x))).$$

It is easy to see that $\tilde{\Psi}$ is a G -equivariant isomorphism, hence it induces an isomorphism Ψ . \square

Globalizations of certain W_n -equivariant \mathcal{O}_{D_n} -modules provide an important source of natural vector bundles. We outline the construction now.

Definition 4.21. Let $Vec(D_n/W_n)$ denote the category of finite rank $\mathbb{C}[[x_1, \dots, x_n]]$ -modules with a compatible $\langle Aut(D_n), W_n \rangle$ -module structure.

Remark 4.22. Equivalently, an object of $Vec(D_n/W_n)$ is a finite rank $\mathbb{C}[[x_1, \dots, x_n]]$ -module M with compatible W_n -action, such that the Euler operator E acts diagonalizably with integer eigenvalues on the associated graded module, $gr(M) = \bigoplus_k \mathfrak{m}_0^k M / \mathfrak{m}_0^{k+1}$.

We note that the globalization functor \mathcal{GL}_X applied to the W_n -module $\mathbb{C}[[x_1, \dots, x_n]]$ yields jets of the structure sheaf \mathcal{O}_X , compatibly with the action of \mathcal{O} on any $M \in Vec(D_n/W_n)$. This endows the globalization $\mathcal{GL}_X(M)$ of $M \in \mathcal{W}_{\mathcal{O}}^{int}$ -mod with the structure of a vector bundle. The naturality of the torsor \mathcal{M}_X under étale morphisms gives a functor:

$$\mathcal{GL} : Vec(D_n/W_n) \rightarrow \mathcal{NVB}_n.$$

Conversely, given a natural vector bundle \mathcal{V} , the evaluation $ev_{D_n} : \mathcal{V} \mapsto \mathcal{V}(D_n)$ is naturally an object of $Vec(D_n/W_n)$.

Proposition 4.23. *The functors ev_{D_n} and \mathcal{GL} define mutually inverse equivalences of categories.*

Proof. A natural isomorphism $\mathcal{GL} \circ ev_{D_n}(\mathcal{V}) \rightarrow \mathcal{V}$ can be constructed as in the proof of 4.20. For $M \in Vec(D_n/W_n)$ -mod, we construct a natural isomorphism $M \rightarrow ev_{D_n} \circ \mathcal{GL}(M)$, as follows. Let $m \in M$. We have a canonical trivialization of the associated bundle $M_{\mathcal{M}_{D_n}}$ over D_n . For $m \in M$, we define a section $f_m \in \Gamma(D_n, \mathcal{GL}_{D_n}(M))$, by:

$$f_m(y_1, \dots, y_n) = e^{\sum_i y_i \partial_i} \cdot m = \sum_I \frac{1}{I!} y^I (\partial_I m).$$

On the right hand side, ∂_i acts on $m \in M$ via the W_n -action, I ranges over all multi-indices $I = (i_1, \dots, i_n)$ with each $i_k \geq 0$, $y^I = y^{i_1} \dots y^{i_n}$, and $\partial_I = \partial_{i_1} \dots \partial_{i_n}$. In other words, f_m is the unique flat section taking value m at the closed point. The correspondence $m \mapsto f_m$ is clearly injective and surjective, since a flat section on D_n is uniquely determined by its value at the origin. □

In summary, we have the following categories, and functors between them:

$$Vec(D_n/W_n) \begin{array}{c} \xrightarrow[\sim]{\mathcal{GL}} \\ \xleftarrow{ev_{D_n}} \end{array} \mathcal{NVB}_n \begin{array}{c} \xrightarrow[\sim]{St} \\ \xleftarrow{Assoc} \end{array} Aut(D_n)\text{-mod}_f \quad (13)$$

Remark 4.24. By Proposition 4.8, the equivalences outlined above reduce the problem of studying natural vector bundles to that of studying finite dimensional representations of the algebraic groups $Aut^k(D_n)$. These groups are unipotent extensions of the group GL_n ; as such their finite dimensional irreducible representations are all pulled back from GL_n .

4.4 Lower central series of locally free algebras

Having reviewed the basic setup for formal geometry and natural vector bundles, we now explain the relation to lower central series.

Lemma 4.25. *Let \widehat{A} be a local ring, non-canonically isomorphic to \widehat{A}_n . Then any isomorphism $\psi : \widehat{A}/M_2 \xrightarrow{\sim} \mathbb{C}[[x_1, \dots, x_n]]$ induces a canonical isomorphism $\widetilde{\psi} : N_k(\widehat{A}) \xrightarrow{\sim} N_k(\widehat{A}_n)$. Moreover, for $g \in \text{Aut}(D_n)$, we have $\widetilde{g \circ \psi} = \widetilde{g} \circ \widetilde{\psi}$.*

Proof. We have the Feigin-Shoikhet natural isomorphisms $\xi : (\widehat{A}/M_3)_\star \cong \Omega^{ev}(\widehat{A}/M_2)$. Recall that $\Omega^{ev}(-)$ is functorial w.r.t isomorphisms: an isomorphism $f : A_1 \xrightarrow{\sim} A_2$ of commutative algebras induces an isomorphism $\Omega^{ev}(f) : \Omega^{ev}(A_1) \xrightarrow{\sim} \Omega^{ev}(A_2)$. Thus, given ψ , we have the natural isomorphism $\overline{\Psi}$, defined as the composition:

$$\begin{array}{ccc} (\widehat{A}/M_3)_\star & \xrightarrow[\xi]{\sim} & \Omega^{ev}(\widehat{A}/M_2) \\ \downarrow \overline{\Psi} & & \downarrow \Omega^{ev}(\psi) \\ (\widehat{A}_n/M_3)_\star & \xleftarrow[\xi^{-1}]{\sim} & \Omega^{ev}(\widehat{A}_n/M_2) \end{array}$$

Let $\Psi : \widehat{A} \rightarrow \widehat{A}_n$ denote an arbitrary isomorphism lifting of $\overline{\Psi}$. We claim that $\Psi|_{N_k(\widehat{A})}$ is independent of the choice of lift. Suppose that Ψ' is another lift; then $\Psi'(a) - \Psi(a) \in M_3(\widehat{A}_n)$, for all $a \in \widehat{A}$. The containments $M_3 M_k \subset M_{k+2}$, and $[A, M_3] \subset L_4$ (Theorem 1.3 from [BJ13a]) imply that Ψ and Ψ' agree modulo $M_{k+1}(\widehat{A}_n)$:

$$\begin{aligned} \Psi(a_0[a_1, [a_2, \dots, [a_{k-1}, a_k]] \dots) &= \Psi(a_0)[\Psi(a_1), \dots [\Psi(a_{k-1}), \Psi(a_k)] \dots] \\ &= \Psi'(a_0)[\Psi'(a_1), \dots [\Psi'(a_{k-1}), \Psi'(a_k)] \dots] \pmod{M_{k+1}}. \end{aligned}$$

Thus, we may set $\widetilde{\psi} := \Psi$. The second part of the claim follows from the naturality in constructing $\overline{\Psi}$. \square

Similarly, any $\chi \in W_n$ induces an endomorphism of $N_k(\widehat{A}_n)$, compatible with the identification $W_n^0 \cong \text{Lie}(\text{Aut}(D_n))$. As a consequence, each component $N_k(\widehat{A}_n)$ is a module for the pair $(\text{Aut}(D_n), W_n)$. Thus, we have the sheaf $\mathcal{GL}_X(N_k(\widehat{A}_n))$ on X . The main result of this section is the following:

Theorem 4.26. *We have $\widetilde{N}_k(A) \cong \mathcal{GL}_X(N_k(\widehat{A}_n))$ as vector bundles on X . In particular, we have $N_k(A) \cong \Gamma(X, \mathcal{GL}_X(N_k(\widehat{A}_n)))$, as $U(\text{Vect}(X)) \ltimes \mathcal{O}_X$ -modules.*

Applying the construction of Proposition 2.6 we may regard $N_k(\widehat{A}_n)$ as an object of $\mathcal{W}_{n, \mathcal{O}}^{int}\text{-mod}$. Thus, combining Theorem 4.26 and the existence of Kapranov's locally free lifts $A(X)$ of any smooth X , we have:

Corollary 4.27. *The assignment $X \mapsto N_k(A(X))$, determines a natural vector bundle.*

Remark 4.28. It should be noted here (see [Kap98], Remark 1.6.4) that the thickenings $A(X)$ constructed by Kapranov are not themselves functorial in X , only the components $N_k(A(X))$ are, by Theorem 4.26.

Proof of Theorem 4.26. Let $p : \mathcal{M}_X \rightarrow X$ denote the canonical projection, and let $U \subset X$ open. For each $x \in X$, we have canonical maps,

$$\rho_x : \widetilde{N}_k(U) \rightarrow (\widetilde{N}_k)_{(\mathfrak{m}_x)} \xrightarrow{\sim} N_k(A_{(\mathfrak{m}_x + M_2)}), \quad (14)$$

composing the restriction to stalks with the isomorphism of Corollary 3.9. We define:

$$\begin{aligned}\varphi : \widetilde{N}_k(U) &\rightarrow \Gamma(p^{-1}(U), \mathcal{M}_X \times N_k(\widehat{A}_n)), \\ \varphi(f)(x, \psi) &= \widetilde{\psi}(\rho_x(f)),\end{aligned}$$

where $\widetilde{\psi}$ is the lift of ψ constructed in Lemma 4.25.

Proposition 4.29. *The section $\varphi(f)$ is \widehat{G} -equivariant, for all $f \in \widetilde{N}_k(U)$.*

Proof. We compute:

$$\begin{aligned}g \cdot (\varphi(f)(x, \psi)) &= g \cdot (\widetilde{\psi}(\rho_x(f))) \\ &= \widetilde{g}(\widetilde{\psi}(\rho_x(f))) \\ &= \widetilde{(g \circ \psi)}(\rho_x(f)) \\ &= \varphi(f)(x, g \circ \psi).\end{aligned}$$

□

By descent, $\varphi(f)$ defines a section in $\Gamma(U, N_X) \cong \Gamma(p^{-1}(U), \mathcal{M}_X \times N_k(\widehat{A}_n))^{\widehat{G}}$, which we also call $\varphi(f)$.

Proposition 4.30. *The section $\varphi(f)$ is flat, for all $f \in \widetilde{N}_k(U)$.*

Proof. A vector field ξ gives rise to a family ξ_x of derivations of $\mathcal{O}(X_{(\mathfrak{m}_x)})$ at each smooth point $x \in X$. By definition of the Harish-Chandra connection, $\phi(f)$ is flat, and only if, the action of ξ_x on each $N_k(A_{(x)})$ agrees with the action of ξ on $N_k(A)_{(x)}$. Thus flatness of $\phi(f)$ follows from the compatibility of the isomorphism of Theorem 3.1 with derivations. □

Thus, the map φ defines a map of sheaves, $\varphi : \widetilde{N}_k(A) \rightarrow \mathcal{GL}_X(N_k(\widehat{A}_n))$. It suffices to check that φ is an isomorphism on stalks, which follows from Corollary 3.9. □

It now follows from Remark 4.24 that the data of the lower central series components N_k , on any locally free algebra A of rank n , are completely determined by the standard fiber of $N_k(A_n)$. This is computationally useful even for $N_k(A_n)$ itself, as the examples below illustrate.

Example 4.31. A basis for the standard fiber $St(N_3(A_n)) = N_3/A_n^+ N_3$ is:

$$\{[x_i, [x_i, x_j]], [x_i, x_j][x_k, x_l]\}.$$

We conclude that $St(N_3(A_2))$ is the W_n^0 -module

$$0 \rightarrow V_{(2,2)} \rightarrow St(N_3(A_2)) \rightarrow V_{(2,1)} \rightarrow 0.$$

The sequence is not split, as $x_2^2 \partial_2[x_1, [x_1, x_2]] = 2[x_1, x_2]^2$, modulo $A_n^+ N_3$. This may be contrasted with [EKM09], Proposition 5.3, which derives a *split* short exact sequence:

$$0 \rightarrow \mathcal{F}_{(2,2)} \rightarrow N_3(A_2) \rightarrow \mathcal{F}_{(2,1)} \rightarrow 0$$

of W_n -modules, where $\mathcal{F}_{(m,n)}$ denotes the tensor field module co-induced from the \mathfrak{gl}_2 -module $V_{(m,n)}$. However, the splitting does not commute with the action of $\mathcal{O}_{\mathbb{A}^2}$.

Example 4.32. In [Ker13], Theorem 1.1, a bound is established on the degree $|\lambda|$ of tensor field modules \mathcal{F}_λ appearing in a W_n composition series of $N_k(A_n)$:

$$|\lambda| \leq \begin{cases} 2k - 2 & k \text{ odd}; \\ 2k - 2 + 2\lfloor \frac{n-2}{2} \rfloor & k \text{ even.} \end{cases}$$

Kerchev's bound can be re-interpreted as a bound on the degree $|\lambda|$ of the \mathfrak{gl}_n -representation V_λ appearing in $St(N_k(A_n))$, which can be established simply by bounding the degree of elements of a spanning set. Using [BJ13a], Corollary 1.5 it is easy to show that $St(N_k(A))$ is spanned by elements of the form,

$$a \star [l_1, [\cdots [l_{k-1}, l_k] \cdots],$$

where l_1, l_k have degree at most one, l_2, \dots, l_{k-1} have degree at most 2, and a is a product of simple brackets $[x_i, x_j]$. If k is odd, then $M_2 L_k \subset M_{k+1}$ in that case, proving the bound. If k is even, a may be non-trivial, but we may assume $a[l_{k-1}]l_k \neq 0 \pmod{M_3}$; otherwise:

$$\begin{aligned} a \star [l_1, [\cdots, [l_{k-1}, l_k] \cdots]] &= [a \star l_1, [\cdots, [l_{k-1}, l_k] \cdots]] \\ &= [l_1, [a \star l_2, [\cdots, [l_{k-1}, l_k] \cdots]] \\ &= \cdots \\ &= [l_1, [\cdots, a \star [l_{k-1}, l_k]] \cdots]. \end{aligned}$$

by repeated application of Kerchev's Lemma 2.1.

References

- [AJ10] N. Arbesfeld and D. Jordan, *New results on the lower central series quotients of a free associative algebra*, Journal of Algebra **323** (2010), no. 6, 1813–1825.
- [BB11] M. Balagović and A. Balasubramanian, *On the lower central series quotients of a graded associative algebra*, J. Algebra **328** (2011), 287–300. MR 2745567 (2012b:16110)
- [BEJ⁺12] S. Bhupatiraju, P. Etingof, D. Jordan, W. Kuzmaul, and J. Li, *Lower central series of a free associative algebra over the integers and finite fields*, J. Algebra **372** (2012), 251–274. MR 2990011
- [BJ13a] A. Bapat and D. Jordan, *Lower central series of free algebras in symmetric tensor categories*, J. Algebra **373** (2013), 299–311. MR 2995028
- [BJ13b] B. Bond and D. Jordan, *The lower central series of the symplectic quotient of a free associative algebra*, J. Pure Appl. Algebra **217** (2013), no. 4, 689–699. MR 2983843
- [BK04] R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in algebraic context*, Moscow Mathematical Journal **4** (2004), no. 3, 559–592.

- [CF13] C. Cordwell and T. Fei, *Lower central series quotients of finitely generated algebras over the integers, in preparation.*
- [CQ95] Joachim Cuntz and Daniel Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289. MR 1303029 (96c:19002)
- [DE08] G. Dobrovolska and P. Etingof, *An upper bound for the lower central series quotients of a free associative algebra*, Int. Math. Res. Not. IMRN (2008), no. 12, Art. ID rnn039, 10. MR 2426755 (2009g:16030)
- [EKM09] P. Etingof, J. Kim, and X. Ma, *On universal lie nilpotent associative algebras*, Journal of Algebra **321** (2009), no. 2, 697–703.
- [FBZ01] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs **88** (2001).
- [FS07] B. Feigin and B. Shoikhet, *On $[A, A]/[A, [A, A]]$ and on a W_n -action on the consecutive commutators of free associative algebras*, Math. Res. Lett. **14** (2007), no. 5, 781–795. MR 2350124 (2009b:16055)
- [Jen47] S. Jennings, *On rings whose associated lie rings are nilpotent*, Bull. Amer. Math. Soc. **53** (1947), 593–597.
- [Kap98] M. Kapranov, *Noncommutative geometry based on commutator expansions*, J. Reine Angew. Math. **505** (1998), 73–118. MR 1662244 (2000b:14003)
- [Ker13] G. Kerchev, *On the filtration of a free algebra by its associative lower central series*, J. Algebra **375** (2013), 322–327. MR 2998959